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Classical Lie algebras and Drinfeld doubles

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Abstract
The Drinfeld double structure underlying the Cartan series A_n, B_n, C_n, D_n of simple Lie algebras is discussed. This structure is determined by two disjoint solvable subalgebras matched by a pairing. For the two nilpotent positive and negative root subalgebras the pairing is natural and in the Cartan subalgebra is defined with the help of a central extension of the algebra. A new completely determined basis is found from the compatibility conditions in the double, and a different perspective for quantization is presented. Other related Drinfeld doubles on $\mathbb{C}$ are also considered.

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1. Introduction

The motivation of this paper is twofold. Firstly, it is well known that several bases are used for the Cartan subalgebra and different authors (Bourbaki, Chevalley, Helgason, Weyl, . . .) introduce different conventions for the commutators of root generators (see for instance [1]). Indeed, there were no objective reasons up till now for a particular choice, we could say that no 'canonical' basis for simple Lie algebras existed up to now. However, we show here that by imposing a Drinfeld double structure on semi-simple Lie algebras the basis is completely determined. Thus, up to an overall factor (i) the Cartan subalgebra basis must be orthonormal and (ii) the normalization of the basis elements must be accomplished and fixed in a way different from the quoted conventions. In conclusion, as a last (small) step of the Cartan programme, a canonical basis is found for all classical Cartan series of simple Lie algebras.

Second, we are involved in a long term programme on quantum deformations of Lie algebras [2]. In this context, the relevance of (quantum) Drinfeld double structures for the construction of simple quantum algebras and groups has been underlined (see [3–5], as well as [6] from the point of view of a generalization of the Iwasawa decomposition). In this respect, the above mentioned canonical basis, related with a co-algebra structure, constitutes the first step for an alternative way to quantize semi-simple Lie algebras [3, 7] by making use of its underlying Drinfeld double structure. In particular, the approach presented here will allow us to consider separately the quantization of the two Borel sub-bialgebras of positive and negative roots, obtaining in this way the whole quantum coproduct, and getting afterwards the deformation of the remaining (crossed) commutation rules.

In [8] (where more technical details can be found), we have presented the results for the Cartan series A_n. In this paper, we show explicitly the canonical bases and the Drinfeld doubles for the remaining series B_n, C_n, D_n of simple Lie algebras.

Let us introduce the notation and basis structure of our approach. A Lie algebra $\bar{g}$ is called a Drinfeld double [3] if it can be endowed with a Manin triple structure [3–5], i.e. a set of three Lie algebras $(s_+, s_-, \bar{g})$ such that s_+ and s_- are disjoint subalgebras of $\bar{g}$ having the same dimension, $\bar{g} = s_+ + s_-$ as vector spaces, and the crossed commutation rules between s_+ and s_- are defined in terms of the structure tensors of s_+ and s_-. More explicitly, let $\{Z_p\}$ and $\{z^p\}$ be bases of s_+ and s_-, respectively, with Lie commutators

$$[Z_p, Z_q] = f^r_{p,q} Z_r, \qquad [z^p, z^q] = c^{p,q}_r z^r. \tag{1.1}$$

Provided that the following relations

$$c^{p,q}_r f^r_{s,t} = c^{p,r}_s f^q_{r,t} + c^{r,q}_s f^p_{r,t} + c^{p,r}_t f^q_{s,r} + c^{r,q}_t f^p_{s,r} \tag{1.2}$$

are fulfilled, a pairing between s_+ and s_- (i.e., a non-degenerate symmetric bilinear form on the vector space $s_+ + s_-$ for which $s_\pm$ are isotropic) can be defined

$$\langle Z_p, Z_q\rangle = 0, \qquad \langle Z_p, z^q\rangle = \delta^q_p, \qquad \langle z^p, z^q\rangle = 0, \tag{1.3}$$

and the remaining commutators of $\bar{g}$ are

$$[z^p, Z_q] = f^p_{q,r} z^r - c^{p,r}_q Z_r. \tag{1.4}$$

An associated Lie bialgebra structure $(\bar{g}, \delta)$ is also determined by the structure tensors of s_+ and s_- that are also Lie sub-bialgebras:

$$\delta(Z_p) = -c^{q,r}_p Z_q \otimes Z_r, \qquad \delta(z^p) = f^p_{q,r} z^q \otimes z^r. \tag{1.5}$$

An important property of the Drinfeld double structure is that the cocommutator (1.5) can be derived from the classical r-matrix $\sum_p z^p \otimes Z_p$, or from its skew-symmetric form

$$r = \frac{1}{2}\sum_p z^p \wedge Z_p. \tag{1.6}$$

Finally, we point out a result that turns out to be a cornerstone for our approach: any Drinfeld double $\bar{g}$ is a Lie algebra with a quadratic Casimir C_D that in a certain basis $\{Z_p, z^p\}$ can be written as

$$C_D = \sum_p [z^p, Z_p]_+, \tag{1.7}$$

where $[z^p, Z_p]_+$ denotes the anticommutator. Therefore, any (even dimensional) Lie algebra with a quadratic Casimir in the form (1.7) is a good candidate for a Drinfeld double.

When one tries to implement the Drinfeld double structure to the Cartan series of simple Lie algebras a difficulty appears [5] for any classical algebra: a Drinfeld double can be built starting from two algebras $s_\pm$ isomorphic to positive and negative Borel subalgebras $b_\pm$. However, $b_\pm$ do have the Cartan subalgebra in common and, therefore, cannot be identified as $s_\pm$.

In [8], the problem has been circumvented for $gl(n)$ by enlarging the algebra in such a way that two disjoint solvable subalgebras $s_\pm$, isomorphic to Borel subalgebras, can be properly paired. Here we follow the same approach for all the Cartan series of semi-simple Lie algebras. Thus,

1. the n-dimensional Cartan subalgebra h_n is enlarged by a central Abelian algebra t_n generated by I_j, $(j = 1, \dots, n)$;
2. a new basis in the $2n$-dimensional Abelian algebra $h_n \oplus t_n$ is defined by
$$X_j := \tfrac{1}{\sqrt{2}}(H_j + \mathbf{i}I_j), \qquad x^j := \tfrac{1}{\sqrt{2}}(H_j - \mathbf{i}I_j), \tag{1.8}$$
where $\mathbf{i}$ is the imaginary unit, and
3. the two disjoint and isomorphic solvable Lie algebras s_+ and s_-, which contain the X_i and x^i generators and the positive and negative roots of g, respectively, define a Weyl–Drinfeld double on $\bar{g} = g \oplus t_n$.

It is worth noting that the Drinfeld double underlying by the classical Lie algebras has the peculiarity that the structure constants $c_r^{p,q}$ and $f_{p,q}^r$ (1.1) verify the relation $c_r^{p,q} = -f_{p,q}^r$. So, we shall say that $\bar{g}$ is a Weyl–Drinfeld double (self-dual in the terminology of [9]).

In order to define such Weyl–Drinfeld double, we find that the usual description of simple Lie algebras in terms of the Chevalley–Cartan basis (and, obviously, Serre relations) is not suitable since, as the Killing form shows, the Cartan subalgebra basis is not orthonormal. In contrast, the Cartan subalgebra basis is orthonormal in the oscillator realization [10, 11] and for that reason we compare our results with this last one. We shall show that the normalization of the root vectors in the Drinfeld doubles is slightly different.

The paper is organized as follows. In section 2, starting from the bosonic and fermionic oscillator realizations of simple Lie algebras we write the suitable bases for the Cartan series. Section 3 presents the four Cartan series in terms of Drinfeld double algebras. The final section is devoted to comment on more general approaches to the Drinfeld double structure of simple Lie algebras.

2. Weyl–Drinfeld basis for classical Lie algebras

2.1. A_n Series

This series is the only one that supports both a bosonic as well as a fermionic oscillator realization. So, in terms of bosonic oscillators $\left([b_i, b_j^\dagger] = \delta_{ij}\right)$ the generators of A_n can be written as
$$H_i := \tfrac{1}{2}\{b_i^\dagger, b_i\}, \qquad F_{ij} := b_i^\dagger b_j, \quad i \neq j. \tag{2.1}$$
By using fermionic oscillators $\left(\{a_i, a_j^\dagger\} = \delta_{ij}\right)$ we have
$$H_i := \tfrac{1}{2}[a_i^\dagger, a_i], \qquad F_{ij} := a_i^\dagger a_j, \quad i \neq j, \tag{2.2}$$
where $i, j = 1, \dots, n+1$. Therefore, we have $n+1$ Cartan generators H_i (remember that $\sum_i H_i$ is an additional central generator) and $n(n+1)$ generators F_{ij} associated with the roots. Definitions (2.1) and (2.2) are slightly different from those given in [10] for a better description of the other series.

In this basis, the explicit commutation rules for A_n are
$$\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, F_{jk}] = (\delta_{ij} - \delta_{ik})F_{jk},\\
&[F_{ij}, F_{kl}] = (\delta_{jk}F_{il} - \delta_{il}F_{kj}) + \delta_{jk}\delta_{il}(H_i - H_j).
\end{aligned} \tag{2.3}$$

2.2. C_n Series

The algebra C_n contains A_{n-1} as a subalgebra and its Weyl–Drinfeld basis is given by the generators H_i, F_{ij} (2.1) together with

$$\begin{aligned} P_{ii} &:= \frac{b_i^\dagger b_i^\dagger}{\sqrt{2}}, \qquad & P_{ij}(=P_{ji}) &:= b_i^\dagger b_j^\dagger, \\ Q_{ii} &:= -\frac{b_i b_i}{\sqrt{2}}, \qquad & Q_{ij}(=Q_{ji}) &:= -b_i b_j, \qquad i < j, \end{aligned} \tag{2.4}$$

where $i, j = 1, \ldots, n$. In this way we have n Cartan generators H_i, $n(n-1)$ generators F_{ij}, two sets of $\frac{1}{2}n(n-1)$ generators P_{ij} and Q_{ij} and, finally, two sets of n generators P_{ii} and Q_{ii} (where the factor $1/\sqrt{2}$, that does not appear in [10], is imposed by the Weyl–Drinfeld double).

The nonvanishing commutation rules for C_n are (2.3) and

$$\begin{aligned} &[H_i, P_{jj}] = 2\delta_{ij} P_{jj}, \qquad [H_i, P_{jk}] = (\delta_{ij} + \delta_{ik}) P_{jk}, \\ &[H_i, Q_{jj}] = -2\delta_{ij} Q_{jj}, \qquad [H_i, Q_{jk}] = -(\delta_{ij} + \delta_{ik}) Q_{jk}, \\ &[F_{ij}, P_{kk}] = \sqrt{2}\delta_{jk} P_{ik}, \qquad [F_{ij}, P_{kl}] = \delta_{jk} P_{il} + \delta_{jl} P_{ik} + \sqrt{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) P_{ii}, \\ &[F_{ij}, Q_{kk}] = -\sqrt{2}\delta_{ik} Q_{jk}, \qquad [F_{ij}, Q_{kl}] = -(\delta_{ik} Q_{jl} + \delta_{il} Q_{jk}) - \sqrt{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) Q_{jj}, \\ &[P_{ii}, Q_{jj}] = 2\delta_{ij} H_i, \quad [P_{ii}, Q_{jk}] = \sqrt{2}(\delta_{ij} F_{ik} + \delta_{ik} F_{ij}), \quad [P_{ij}, Q_{kk}] = \sqrt{2}(\delta_{ik} F_{jk} + \delta_{jk} F_{ik}), \\ &[P_{ij}, Q_{kl}] = (\delta_{ik} F_{jl} + \delta_{jl} F_{ik} + \delta_{jk} F_{il} + \delta_{il} F_{jk}) + (\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})(H_i + H_j). \end{aligned} \tag{2.5}$$

2.3. D_n Series

As in the preceding case, D_n also contains A_{n-1} as a subalgebra. However, now D_n admits a fermionic oscillator realization. Besides the generators H_i, F_{ij} (2.2) we need two more sets of $\frac{1}{2}n(n-1)$ generators S_{ij} and T_{ij} given by $(i, j = 1, \ldots, n)$

$$S_{ij}(=-S_{ji}) := a_i^\dagger a_j^\dagger, \qquad T_{ij}(=-T_{ji}) := -a_i a_j, \quad i < j, \tag{2.6}$$

that completes the realization of the D_n algebra in the Weyl–Drinfeld basis.

The nonvanishing commutator rules for D_n are (2.3) together with

$$\begin{aligned} &[H_i, S_{jk}] = (\delta_{ij} + \delta_{ik}) S_{jk}, \qquad [H_i, T_{jk}] = -(\delta_{ij} + \delta_{ik}) T_{jk}, \\ &[F_{ij}, S_{kl}] = \delta_{jk} S_{il} - \delta_{jl} S_{ik}, \qquad [F_{ij}, T_{kl}] = -\delta_{ik} T_{jl} + \delta_{il} T_{jk}, \\ &[S_{ij}, T_{kl}] = (-\delta_{jk} F_{il} - \delta_{il} F_{jk} + \delta_{ik} F_{jl} + \delta_{jl} F_{ik}) + (\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})(H_i + H_j). \end{aligned} \tag{2.7}$$

2.4. B_n Series

The Lie algebra B_n contains D_n as a subalgebra. Hence, the basis of B_n in the fermionic realization will be composed by generators (2.2), (2.6) together with $2n$ additional generators

$$U_i := \tfrac{1}{\sqrt{2}} a_i^\dagger, \qquad V_i := \tfrac{1}{\sqrt{2}} a_i, \tag{2.8}$$

where, again, the factor $1/\sqrt{2}$ is required by the Weyl–Drinfeld double. The commutation rules for B_n are (2.3), (2.7) plus the non-vanishing relations involving U and V:

$$\begin{aligned} &[H_i, U_j] = \delta_{ij} U_j, && [H_i, V_j] = -\delta_{ij} V_j, \\ &[F_{ij}, U_k] = \delta_{jk} U_i, && [F_{ij}, V_k] = -\delta_{ik} V_j, \\ &[T_{ij}, U_k] = \delta_{ik} V_j - \delta_{jk} V_i, && [S_{ij}, V_k] = -\delta_{ik} U_j + \delta_{jk} U_i, \\ &[U_i, U_j] = S_{ij}, && [U_i, V_j] = (1-\delta_{ij}) F_{ij} + \delta_{ij} H_i, \\ &[V_i, V_j] = -T_{ij}. && \end{aligned} \tag{2.9}$$

3. Weyl–Drinfeld doubles and Lie bialgebras

3.1. A_n Series

We will follow [8]. Let us consider the Lie algebra $gl(n+1) = A_n \oplus h$, where h is the Lie algebra generated by $\sum H_i$.

We introduce $n+1$ central generators I_i and define the new generators X_i and x^i in terms of the H_i and I_i as follows:

$$X_i := \tfrac{1}{\sqrt{2}}(H_i + \mathbf{i} I_i), \qquad x^i := \tfrac{1}{\sqrt{2}}(H_i - \mathbf{i} I_i). \tag{3.1}$$

Now we consider two solvable Lie algebras s_+ and s_- with dimension $(n+1)(n+2)/2$ and defined by

$$\begin{aligned} &s_+: \quad \{X_i, F_{ij}\}, \qquad i, j = 1, \ldots, n+1, \quad i < j, \\ &s_-: \quad \{x^i, f^{ij}\}, \qquad i, j = 1, \ldots, n+1, \quad i < j, \end{aligned}$$

where $f^{ij} := F_{ji} (i < j)$. Note that $gl(n+1) \oplus t_{n+1} = s_+ + s_-$ as vector spaces, being t_{n+1} the Abelian Lie algebra generated by the I_i. The commutation rules for s_+ and s_- are

$$\begin{aligned} &[X_i, X_j] = 0, && [X_i, F_{jk}] = \tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik}) F_{jk}, && [F_{ij}, F_{kl}] = (\delta_{jk} F_{il} - \delta_{il} F_{kj}), \\ &[x^i, x^j] = 0, && [x^i, f^{jk}] = -\tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik}) f^{jk}, && [f^{ij}, f^{kl}] = -(\delta_{jk} f^{il} - \delta_{il} f^{kj}). \end{aligned}$$

Assuming that the two algebras s_+ and s_- are paired by

$$\langle x^i, X_j \rangle = \delta^i_j, \qquad \langle f^{ij}, F_{kl} \rangle = \delta^i_k \delta^j_l, \tag{3.2}$$

we can define a bilinear form on the vector space $s_+ + s_-$ in terms of (3.2) such that both $s_\pm$ are isotropic.

Taking into account (1.4) one can easily write the crossed commutation rules, and the compatibility relations (1.2) can be also checked. Hence, we obtain the Lie algebra $gl(n+1) \oplus t_{n+1}$, whose commutation rules in the initial basis $\{H_i, F_{ij}, I_i\}$ are given in (2.3) plus $[I_i, \cdot] = 0$. Thus, $(s_+, s_-, gl(n+1) \oplus t_{n+1})$ is a Manin triple.

The canonical Lie bialgebra structure for $gl(n+1) \oplus t_{n+1}$ is determined by the cocommutator δ (1.5) and reads

$$\begin{aligned} &\delta(I_i) = 0, \qquad \delta(H_i) = 0, \\ &\delta(F_{ij}) = -\frac{1}{2} F_{ij} \wedge (H_i - H_j) - \frac{\mathbf{i}}{2} F_{ij} \wedge (I_i - I_j) + \sum_{k=i+1}^{j-1} F_{ik} \wedge F_{kj}, \qquad i < j, \\ &\delta(F_{ij}) = \frac{1}{2} F_{ij} \wedge (H_i - H_j) - \frac{\mathbf{i}}{2} F_{ij} \wedge (I_i - I_j) - \sum_{k=j+1}^{i-1} F_{ik} \wedge F_{kj}, \qquad i > j. \end{aligned} \tag{3.3}$$

Note that s_+ and its dual s_- are Lie sub-bialgebras, while A_n is a subalgebra but not a sub-bialgebra.

The classical r-matrix (1.6) is given by

$$r = \frac{1}{2}\sum_{i<j} F_{ji} \wedge F_{ij} + \frac{\mathbf{i}}{2}\sum_{i} H_i \wedge I_i = r_s + r_t.$$

The term r_s generates the standard deformation of $gl(n+1)$ and r_t is a twist (not of Reshetikhin type [12]). When all the I_i s are equal the twist r_t becomes trivial.

It is worth noting that in this construction the chain $gl(n) \oplus t_n \subset gl(n+1) \oplus t_{n+1}$ is preserved at the level of Lie bialgebras.

3.2. C_n Series

The two subalgebras $s\pm$, isomorphic to the Borel subalgebras $b\pm$ of C_n, are

$$s_+:\quad \{X_i, F_{ij}, P_{ij}, P_{ii}\}, \qquad i, j = 1, \ldots, n, \quad i < j,$$
$$s_-:\quad \{x^i, f^{ij}, p^{ij}, p^{ii}\}, \qquad i, j = 1, \ldots, n, \quad i < j,$$

where the X_i, x^i, F_{ij} and f^{ij} were defined in the preceding section and $p^{ij} := Q_{ij}$ and $p^{ii} := Q_{ii}$. The Lie commutators of the subalgebras s_+ and s_- are obtained from (2.3) and (2.5).

The pairing

$$\langle x^i, X_j\rangle = \delta^i_j, \qquad \langle f^{ij}, F_{kl}\rangle = \delta^i_k\delta^j_l, \qquad \langle p^{ij}, P_{kl}\rangle = \delta^i_k\delta^j_l, \qquad \langle p^{ii}, P_{jj}\rangle = \delta^i_j.$$

allows us to derive the crossed commutators (1.4) in this case. It can be proven that the full set of relations are nothing but the commutation rules for $C_n \oplus t_n$. Therefore, we can avoid to check the compatibility conditions (1.2) since $C_n \oplus t_n$ is a well-known Lie algebra, and we conclude that $(s_+, s_-, C_n \oplus t_n)$ is a Manin triple.

The cocommutator δ (1.5) for the generators H_i, F_{ij} is given by (3.3) and for the remaining generators read

$$\delta(P_{ii}) = (H_i + \mathbf{i}I_i) \wedge P_{ii} + \sqrt{2}\sum_{k>i} F_{ik} \wedge P_{ik},$$

$$\delta(P_{ij}) = \frac{1}{2}[(H_i + H_j) + \mathbf{i}(I_i + I_j)] \wedge P_{ij} + \sqrt{2}F_{ij} \wedge P_{jj} + \sum_{m>i, m\neq j} F_{im} \wedge P_{mj}, \qquad i < j,$$

$$\delta(Q_{ii}) = (H_i - \mathbf{i}I_i) \wedge P_{ii} + \sqrt{2}\sum_{k>i} F_{ki} \wedge Q_{ik},$$

$$\delta(Q_{ij}) = \frac{1}{2}[(H_i + H_j) - \mathbf{i}(I_i + I_j)] \wedge Q_{ij} + \sqrt{2}F_{ji} \wedge Q_{jj} + \sum_{m>i, m\neq j} F_{mi} \wedge Q_{mj}.$$

As in the previous case, $s_\pm$ are Lie sub-bialgebras.

The r-matrix (1.6), in the basis $\{H_i, F_{ij}, I_i, P_{ij}, Q_{ij}\}$, is written as

$$r = \frac{1}{2}\left(\sum_{i<j} F_{ji} \wedge F_{ij} + \sum_{i\leqslant j} Q_{ij} \wedge P_{ij}\right) + \frac{\mathbf{i}}{2}\sum_{i} H_i \wedge I_i = r_s + r_t.$$

Again r_s generates the standard deformation of C_n and r_t is a twist. When all the I_i are zero r_t vanishes.

Note also that the chain $C_n \oplus t_n \subset C_{n+1} \oplus t_{n+1}$ is preserved at the level of Lie bialgebras.

3.3. D_n Series

The subalgebras s_+ and s_- are

$$
\begin{aligned}
s_+&: \quad \{X_i, F_{ij}, S_{ij}\}, \qquad i, j = 1, \ldots, n, \quad i < j,\\
s_-&: \quad \{x^i, f^{ij}, s^{ij}\}, \qquad i, j = 1, \ldots, n, \quad i < j,
\end{aligned}
$$

where the X_i, x^i and f^{ij} were defined in the preceding sections and with $s^{ij} := T_{ij}$. The commutators of s_+ and s_- are obtained from (2.3) and (2.7).

The crossed commutators (1.4) are obtained by making use of the pairing

$$
\langle x^i, X_j\rangle = \delta^i_j, \qquad \langle f^{ij}, F_{kl}\rangle = \delta^i_k \delta^j_l, \qquad \langle s^{ij}, S_{kl}\rangle = \delta^i_k \delta^j_l.
$$

The complete set of Lie commutators allows us to identify $s_+ + s_-$ with the Lie algebra $D_n \oplus t_n$, hence we can avoid to check the compatibility conditions (1.2). So, $(s_+, s_-, D_n \oplus t_n)$ is a Manin triple.

The cocommutator δ (1.5) for the generators H_i, F_{ij} is given by (3.3) and for the remaining generators takes the form

$$
\begin{aligned}
\delta(S_{ij}) &= \frac{1}{2}[(H_i + H_j) + \mathbf{i}(I_i + I_j)] \wedge S_{ij} + \sum_{k>i, k\neq j} F_{ik} \wedge S_{kj}, \qquad i < j,\\
\delta(T_{ij}) &= \frac{1}{2}[(H_i + H_j) - \mathbf{i}(I_i + I_j)] \wedge T_{ij} + \sum_{k>i, k\neq j} F_{ki} \wedge T_{kj}, \qquad i < j.
\end{aligned}
\tag{3.4}
$$

The subalgebras $s_\pm$ are Lie sub-bialgebras. The chain $D_n \oplus t_n \subset D_{n+1} \oplus t_{n+1}$ is again preserved at the level of Lie bialgebras.

The classical r-matrix (1.6), is given by

$$
r = \frac{1}{2}\left(\sum_{i<j} F_{ji} \wedge F_{ij} + \sum_{i<j} T_{ij} \wedge S_{ij}\right) + \frac{\mathbf{i}}{2}\sum_i H_i \wedge I_i = r_s + r_t,
$$

where r_s generates the standard deformation of D_n and r_t is a twist that vanishes when all the $I_i \to 0$.

3.4. B_n Series

The subalgebras s_+ and s_- are

$$
\begin{aligned}
s_+&: \quad \{X_i, F_{ij}, S_{ij}, U_i\}, \qquad i, j = 1, \ldots, n, \quad i < j,\\
s_-&: \quad \{x^i, f^{ij}, s^{ij}, u^i\}, \qquad i, j = 1, \ldots, n, \quad i < j,
\end{aligned}
$$

where $s^{ij} := T_{ij}$ and $u^i := V_i$. Their commutators are obtained from (2.3) and (2.9).

The appropriate pairing for this case is

$$
\langle x^i, X_j\rangle = \delta^i_j, \qquad \langle y^{ij}, Y_{kl}\rangle = \delta^i_k \delta^j_l, \qquad \langle s^{ij}, S_{kl}\rangle = \delta^i_k \delta^j_l, \qquad \langle u^i, U_j\rangle = \delta^i_j.
$$

The complete set of Lie commutators allows us to identify $s_+ + s_-$ with the Lie algebra $B_n \oplus t_n$, so we can avoid to check the compatibility conditions (1.2). Hence, $(s_+, s_-, B_n \oplus t_n)$ is a Manin triple.

The cocommutator δ (1.5) for the generators H_i, F_{ij}, S_{ij}, T_{ij} is given by (3.3) and (3.4) and for the remaining generators reads

$$\delta(S_{ij}) = \frac{1}{2}[(H_i + H_j) + \mathbf{i}(I_i + I_j)] \wedge S_{ij} + \sum_{k>i,k\neq j} F_{ik} \wedge S_{kj} + U_i \wedge U_j, \qquad i < j,$$

$$\delta(T_{ij}) = \frac{1}{2}[(H_i + H_j) - \mathbf{i}(I_i + I_j)] \wedge T_{ij} + \sum_{k>i,k\neq j} F_{ki} \wedge T_{kj} + V_i \wedge V_j, \qquad i < j,$$

$$\delta(U_i) = \frac{1}{2}(H_i + \mathbf{i}I_i) \wedge U_i + \sum_{i<k\leqslant n} F_{ik} \wedge U_k,$$

$$\delta(V_i) = \frac{1}{2}(H_i - \mathbf{i}I_i) \wedge V_i + \sum_{1\leqslant k<i} F_{ki} \wedge V_k.$$

Also in this case $s_\pm$ are Lie sub-bialgebras. The chain $B_n \oplus t_n \subset B_{n+1} \oplus t_{n+1}$ is preserved at the level of Lie bialgebras. Note that, while $D_n \subset B_n$ as algebras, $D_n \oplus t_n \not\subset B_n \oplus t_n$ as bialgebras.

The classical r-matrix (1.6) is written as

$$r = \frac{1}{2}\left(\sum_{i<j} F_{ji} \wedge F_{ij} + \sum_{i<j} T_{ij} \wedge S_{ij} + \sum_{i} V_i \wedge U_i\right) + \frac{\mathbf{i}}{2}\sum_{i} H_i \wedge I_i = r_s + r_t.$$

Again r_s generates the standard deformation of B_n and r_t is the twist.

4. Other deformations in $\mathbb{C}$ and real forms

The canonical Weyl–Drinfeld double structures above introduced are not the only possible ones. Indeed, we can consider other possibilities by taking into account that the quadratic Casimir, in the Cartan–Drinfeld basis, can be always written as

$$\mathcal{C}_2 = \sum H_i^2 + \sum \left[X_j^+, X_j^-\right]_+, \tag{4.1}$$

where the H_i determine a well-defined basis in the Cartan subalgebra and X_j^+ (resp. X_j^-) constitutes an appropriate basis for the nilpotent algebra of positive (respectively negative) root generators. Obviously, the roots X_j^+ (respectively X_j^-) can be immediately associated with the subalgebra generated by the Z_p (respectively z^p). Therefore, the problem is to fit the Cartan generators within the scheme.

In the previous section we have given a solution to this problem for any semi-simple Lie algebra, by enlarging the (n dimensional) Cartan subalgebra through the addition of n central generators $\{I_j\}$ and by taking into account that

$$\sum_i^n \left(H_i^2 + I_i^2\right) = \sum_i^n \left[\frac{1}{\sqrt{2}}(H_i + \mathbf{i}I_i), \frac{1}{\sqrt{2}}(H_i - \mathbf{i}I_i)\right]_+. \tag{4.2}$$

At this point, the full Drinfeld double structure comes out in a natural way by including the $\frac{1}{\sqrt{2}}(H_i \pm \mathbf{i}I_i)$ generators in the $s_\pm$ subalgebras

$$s_\pm = \left\{\tfrac{1}{\sqrt{2}}(H_i \pm \mathbf{i}I_i), X_j^\pm\right\}.$$

For the Lie algebra A_1 this is the only possible solution. However, for Lie algebras whose rank is even, it is possible to construct a different Drinfeld double without introducing any additional central operator I_i. For instance, in the case of A_2, by using the Gell–Mann basis [13], the two solvable algebras can be chosen as

$$s_{\pm} = \{\lambda_3 \pm \mathbf{i}\lambda_8, \lambda_1 \pm \mathbf{i}\lambda_2, \lambda_4 \pm \mathbf{i}\lambda_5, \lambda_6 \pm \mathbf{i}\lambda_7\}.$$

Another example of double structure can be constructed for the $D_2 \approx A_1 \oplus A_1$ algebra through the solvable algebras

$$s_{\pm} = \{J_3 \pm \mathbf{i}L_3, J_1 \pm \mathbf{i}J_2, L_1 \pm \mathbf{i}L_2\},$$

where the generators J_i belong to the first A_1 algebra and the L_i to the second one.

However, for odd-dimensional algebras at least one additional generator must be introduced in order to get a global even dimension. In general, all the intermediate cases among the canonical case and the previous ones can be considered by introducing an algebra t_m of central elements I_i with $0 \leqslant m \leqslant n$ and such that $(n-m)/2 \in Z^+$. Thus, the two solvable algebras can be defined as

$$s_{\pm} = \left\{\tfrac{1}{\sqrt{2}}(H_i \pm \mathbf{i}H_j), \tfrac{1}{\sqrt{2}}(H_k \pm \mathbf{i}I_k), X_l^{\pm}\right\},$$

where each Abelian subalgebra is constituted by $(n-m)/2$ generators without I_k and m generators containing I_k. If $m < n$ the algebra does not exist for the real field but, in any case, the basis can be constructed in such a way that $c_r^{p,q} = -\bar{f}_{p,q}^{\,r}$. In some sense all these cases could, thus, be considered as Weyl–Drinfeld doubles on $\mathbb{C}$.

Note that in all the previous expressions the chosen normalization of the generators of $s_{\pm}$ could be changed without destroying the bialgebra structure, provided an appropriate inverse factor is introduced in $s_{\mp}$. However, in that case the Weyl–Drinfeld property of the double would be broken.

Summarizing the work, we could say that, in the spirit of Cartan, the classical series of Lie algebras can be considered as, essentially, the Drinfeld doubles associated with self-dual Lie bialgebra structures on a very specific set of solvable Lie algebras (the ones defined by the Cartan generators and the positive roots).

From the perspective of physical applications the interest is focused on real forms. We have shown that simple Lie algebras can be endowed with a Drinfeld double structure by adding a suitable pure imaginary central extension. However, for each classical Lie algebra, if we consider the representation $I_i = 0$ of the extension (1.8), the resulting Lie bialgebra structure is real (and the same will happen with its corresponding quantum deformation). At this point, the standard $*$-involution machinery [1, 5] will allow us to obtain the remaining (quantum) real forms.

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